EEL 711/ELL 711 Minor Test I Semester I 2015-2016

Answer all questions (Marks: Q.1: 20, Q.2: 6, Q.3: 14) Full Marks: 40

1. A 200 MHz frequency band having the range 800 MHz $\leq f <$ 1000 MHz, where f denotes the frequency, is divided into n sub-bands ($n = 2m$, where m is a positive integer) having equal bandwidth of $200/n$ MHz. The ith sub-band has the range

$$\left(800 + \frac{200(i-1)}{n}\right) \text{ MHz} \leq f < \left(800 + \frac{200i}{n}\right) \text{ MHz}, \quad i = 1, \ldots, n.$$

The *probability that the ith sub-band is occupied by some user is* p, where $1/2 \leq p < 1$, and all n sub-bands can be occupied independently. Let X denote the number of occupied sub-bands. It is given that $\Pr\left[\left|X - (n - X)\right| = 2\right] = 2\Pr\left[\left|X - (n - X)\right| = 0\right]$.

(a) Find p in terms of m. [4]

(b) When $\mathbf{E}[X] = 10\sqrt{\mathrm{var}(X)}$: (i) calculate n and p, (ii) calculate the root mean square value of the unoccupied bandwidth. [6+4]

(c) If the frequency band has sub-bands of negligible bandwidth (that is, the number of sub-bands is very large), then, for the value of p obtained in (b)(i), calculate the *root mean square value of the number of occupied sub-bands to be traversed* (in ascending order of frequency) for the 2nd *unoccupied* sub-band to appear. [6]

2. Let $X_1, \ldots, X_{2560}$ be i.i.d. Bernoulli distributed random variables, each with mean $< 1/2$, standard deviation $= \sqrt{1023}/1024$. Calculate $\Pr\left[\sum_{k=1}^{2560} X_k \geq 3\right]$ using the Poisson approximation. [6]

3. A bivariate Gaussian p.d.f. is given by

$$f_{\underline{X}}(\underline{x}) = f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{5}} \exp\left\{-\frac{1}{10}\left[3x_1^2 + 2x_2^2 + 18x_1 + 16x_2 + 2x_1x_2 + 42\right]\right\},$$
$$-\infty < x_1, x_2 < \infty,$$

where $\underline{X} = [X_1 \;\; X_2]^T$, $\underline{x} = [x_1 \;\; x_2]^T$, and $\underline{X} \sim \mathcal{N}(\underline{\mu}, \underline{K})$.

(a) Find the covariance matrix $\underline{K}$ and the mean vector $\underline{\mu}$ of $\underline{X}$. [8]

(b) Calculate $\mathbf{E}\left[(X_2 + 2)^3\right]$ and $\mathbf{E}\left[(X_1 - 1)^4\right]$. [6]

Some Formulae

- Binomial distribution: $\binom{n}{k} p^k (1-p)^{n-k}$, $0 \leq k \leq n$, mean $= np$, variance $= np(1-p)$

- Negative binomial distribution: $\binom{k-1}{r-1} p^r (1-p)^{k-r}$, $r \leq k < \infty$

- If $Y \sim \mathcal{N}(0, 1)$, then

$$\text{p.d.f. } f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad -\infty < y < \infty, \qquad \mathbf{E}\left[Y^{2\ell}\right] = \frac{(2\ell)!}{\ell! 2^\ell}, \quad \ell = 1, 2, 3, \ldots$$

- If $\underline{X} \sim \mathcal{N}(\underline{\mu}, \underline{K})$ and $\underline{X}$ is $L \times 1$, then

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{L/2} \{\det(\underline{K})\}^{1/2}} \exp\left\{-\frac{1}{2}(\underline{x} - \underline{\mu})^T \underline{K}^{-1} (\underline{x} - \underline{\mu})\right\}, \quad \underline{x} \in \mathcal{R}^L$$